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NONLINEAR WAVES IN AN ACTIVE-DISSIPATIVE DISPERSE MEDIUM

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Nonlinear waves in a medium involving dissipation, dispersion, and enhancement described by the generalized Kuramoto–Sivashinsky equation are discussed. Analytical solutions of the equation are obtained in the form of solitary waves. For numerical modeling of the nonlinear waves a difference scheme is suggested. Interaction of nonlinear waves described by the Kuramoto–Sivashinsky model is considered. It is shown that for specified values of the problem parameters there is one solitary wave described by the initial model. The dependences of the velocity and amplitude of this wave on the problem parameters are determined.

Introduction. One of the common nonlinear models used in describing wave processes is the model based on the generalized Kuramoto–Sivashinsky equation

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0 . \tag{1}$$

This equation is used to investigate long waves in the flow of a thin liquid layer down an inclined plane [1, 2] and thermocapillary convection in thin liquid layers [3] and to describe the processes of instability and generation of turbulence in combustion [4].

In the case $\beta = 0$, Eq. (1) is one of the simplest relations describing turbulent processes in an active dissipative medium, and therefore it has been actively investigated in the last several years [5, 6].

For $\alpha > 0, \gamma > 0$ the term with the second derivative corresponds to pumping energy into the system, and the components with the third and fourth derivatives characterize its dispersion and dissipation, respectively.

Below we present some analytical solutions of Eq. (1), a difference scheme for mathematical modeling of the nonlinear waves described by Eq. (1), and results of mathematical modeling of nonlinear waves.

Analytical Solutions of Eq. (1). Equation (1) cannot be integrated by the method of the inverse scattering problem since it does not satisfy the Painleve property [7-9]; however it has some particular solutions. We introduce dimensionless variables in Eq. (1), setting

$$u = \alpha \sqrt{\alpha/\gamma}\, u' , \quad x = \sqrt{\gamma/\alpha}\, x' , \quad t = (\gamma/\alpha^2)\, t' , \quad \sigma = \beta/\sqrt{\alpha\gamma} .$$

Then Eq. (1) acquires the form

$$u_t + uu_x + u_{xx} + \sigma u_{xxx} + u_{xxxx} = 0 \tag{2}$$

(the primes at the variables in (2) are omitted).

We note that Eq. (2) is invariant relative to the substitution

$$u \to -u , \; x \to -x , \; \sigma \to -\sigma , \tag{3}$$

and therefore it will be considered below only for $\sigma \geq 0$.

We will seek a solution of Eq. (2) in the form

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$$u(x,t) = a_0 Y^p + a_1 Y^{p-1} + \dots + a_p , \tag{4}$$

where $Y(x, t)$ satisfies the system of equations [10]

$$Y_x = -Y^2 - \frac{S}{2}, \tag{5}$$

$$Y_t = CY^2 - C_x Y + \frac{1}{2}(SC + C_{xx}), \tag{6}$$

and the variables C and S satisfy the consistency condition [10]

$$S_t + C_{xxx} + 2C_x S + CS_x = 0 . \tag{7}$$

With Eq. (5) taken into consideration, after substitution of $u \cong a_0 Y^p$ into Eq. (2) we obtain $p = 3$, $a_0 = 120$. Thus, a solution of (2) is sought in the form

$$u = 120Y^3 + a_1 Y^2 + a_2 Y + a_3 . \tag{8}$$

Substituting (8) into (2) and equating coefficients of the same power of Y to zero, we arrive at

$$a_1 = -15\sigma , \tag{9}$$

$$a_2 = 60S + \frac{15}{76}(16 - \sigma^2), \tag{10}$$

$$a_3 = -15S_x - 5\sigma S + C + \frac{\sigma}{76}\left(7 - \frac{13}{8}\sigma^2\right). \tag{11}$$

Furthermore, we obtain a system of equations in S, C, and σ as a parameter:

$$-C_x + 3S_{xx} + \frac{5}{4}\sigma S_x + 2S^2 + \frac{5}{152}(16 - \sigma^2)S - \frac{1}{722}\left(\frac{131}{64}\sigma^4 - \frac{87}{8}\sigma^2 + 11\right) = 0 , \tag{12}$$

$$S_{xxx} + \frac{3}{8}\sigma S_{xx} + 3SS_x + \frac{5}{608}(16 - \sigma^2)S_x + \frac{\sigma}{4}S^2 - \sigma\left[\frac{3}{152}\left(\sigma^2 - \frac{10}{3}\right)\right]^2 = 0 , \tag{13}$$

$$S_t + S_{xxxx} + \frac{\sigma}{2}S_{xxx} - 2SS_{xx} + \frac{5}{304}(16 - \sigma^2)S_{xx} - 2S_x^2 + (C - \sigma S)S_x + 2SC_x = 0 , \tag{14}$$

$$\begin{aligned} &C_t + C_x C - 135S_{xx}S_x + \frac{15}{2}\sigma S_{xx}S + \frac{\sigma}{152}\left(\frac{1011\sigma^2}{64} - 39\right)S_{xx} - \frac{225}{8}\sigma S_x^2 - \\ &- 15S_x S^2 + \frac{15}{38}\left(\frac{165}{32}\sigma^2 - 1\right)S_x S + 15C_x S_x + \frac{15}{1444}\left(\frac{1991}{2048}\sigma^4 - \frac{581}{128}\sigma^2 + 17\right)S_x + \\ &+ \frac{45}{4}\sigma S^3 + \frac{3\sigma}{152}\left(\frac{89}{32}\sigma^2 + 3\right)S^2 - 5\sigma\left[\frac{9}{152}\left(\sigma^2 - \frac{10}{3}\right)\right]^2 S + \\ &+ \frac{\sigma}{152^3}\left(\frac{257}{2}\sigma^6 - \frac{16003}{2}\sigma^4 + \frac{97037}{2}\sigma^2 - 77328\right) = 0 . \end{aligned} \tag{15}$$

TABLE 1. Solutions of System of Equations (17)-(19)

σ	0	0	$12/\sqrt{47}$	$16/\sqrt{73}$	4	4
S	$-11/38$	$1/38$	$-1/94$	$-1/146$	$-1/2$	$1/2$

TABLE 2. Analytical Solutions of Eq. (2), $\xi = k/2(x - Ct)$

σ	k	Analytical solution
0	$\sqrt{11/19}$	$C + 15k^3\tanh(\xi)(\tanh^2(\xi) - 9/11)$
0	$1/\sqrt{19}$	$C + 15k^3\tan(\xi)(\tan^2(\xi) + 3)$
$12/\sqrt{47}$	$1/\sqrt{47}$	$C + 15k^3\{[\tanh(\xi) - 1]^3 + 4\}$
$16/\sqrt{73}$	$1/\sqrt{73}$	$C + 15k^3\{\tanh(\xi)[\tanh^2(\xi) + 5] + 4\cosh^2(\xi)\}$
4	1	$C + 4 - 15k^3[1 + \tan(\xi)]\cos^2(\xi)$
4	1	$C - 6 + 15k^3[1 - \tanh(\xi)]\cosh^2(\xi)$

The last system of equations is overdetermined in the general case; however it is consistent for some classes of functions. Setting in (12)-(15)

$$S_x = C_x = S_t = C_t = 0\,, \tag{16}$$

we obtain the following system of algebraic equations:

$$2S^2 + \frac{5}{152}(16 - \sigma^2)\,S - \frac{1}{722}\left(\frac{131}{64}\sigma^4 - \frac{87}{8}\sigma^2 + 11\right) = 0\,, \tag{17}$$

$$\frac{\sigma}{4}S^2 - \sigma\left[\frac{3}{152}\left(\sigma^2 - \frac{10}{3}\right)\right]^2 = 0\,, \tag{18}$$

$$\frac{45}{4}\sigma S^3 + \frac{3\sigma}{152}\left(\frac{89}{32}\sigma^2 + 3\right)S^2 - 5\sigma\left[\frac{9}{152}\left(\sigma^2 - \frac{10}{3}\right)\right]^2 S + \\ + \frac{\sigma}{152^3}\left(\frac{257}{2}\sigma^6 - \frac{16003}{2}\sigma^4 + \frac{97037}{2}\sigma^2 - 77328\right) = 0\,, \tag{19}$$

solutions of which for $C = \text{const}$ are given in Table 1. System (5), (6) is transformed to a linear form by the substitution $Y = \Psi_x/\Psi$:

$$\Psi_{xx} + \frac{S}{2}\Psi = 0\,, \tag{20}$$

$$\Psi_t + C\Psi_x - \frac{C_x}{2}\Psi = 0\,, \tag{21}$$

which for conditions (16) gives the solution

$$\Psi(x, t) = C_1\exp\left\{\frac{k}{2}(x - Ct)\right\} + C_2\exp\left\{-\frac{k}{2}(x - Ct)\right\}, \tag{22}$$

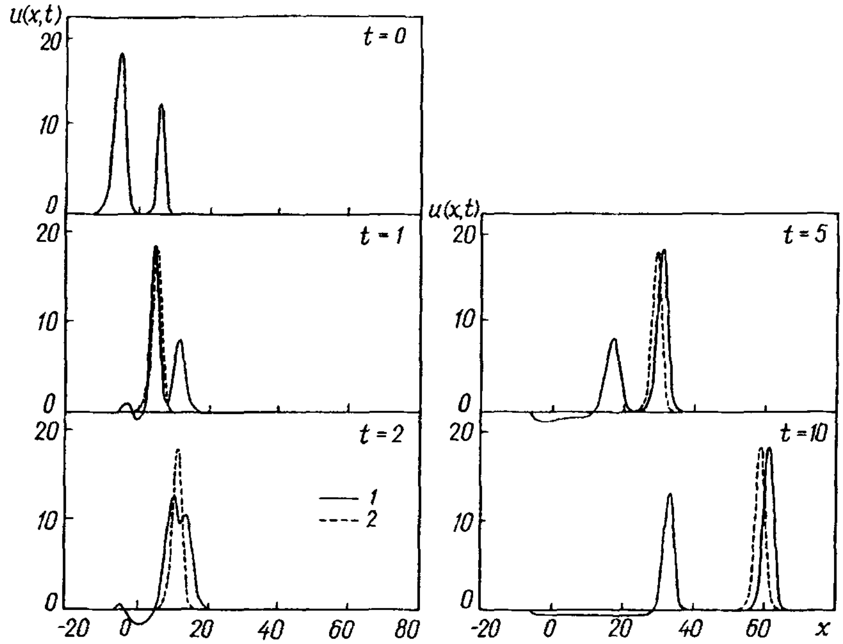

Fig. 1. Interaction of solitary wave (27) with a disturbance having the form $u(x) = 15\cosh^{-2}\{8(x-3)\}$ at the initial moment: 1) interaction with the disturbance; 2) analytical solution without interaction.

where $k^2 = -2S$; C_1 and C_2 are arbitrary constants.

Passing from the function $\Psi(x, t)$ in the form (22) to $Y = \Psi_x/\Psi$, we arrive at the following solution Eq. (2):

$$u = 120Y^3 - 15\sigma Y^2 + 15\left(4S + \frac{1}{76}(16-\sigma^2)\right)Y - 5\sigma S + \frac{\sigma}{76}\left(7 - \frac{13}{8}\sigma^2\right) + C\,, \tag{23}$$

where $Y = (k/2)\tanh\{k/2(x - Ct) + \varphi_0\}$, C and φ_0 are arbitrary constants, and σ and S are given in Table 1.

Formula (23) reflects the fact that Eq. (2) is invariant relative to Galilean transformations

$$(u, x, t) \to (u + C, x - Ct, t)\,. \tag{24}$$

Table 2 gives solutions of Eq. (2) for particular values of σ and $k = -\sqrt{2|S|}$ and $\varphi_0 = 0$.

From formula (23) we can find the asymptotic form of the solution obtained:

$$u \to C + \frac{\sigma}{76}\left(7 - \frac{13}{8}\sigma^2\right) - \frac{5}{4}\sigma k^2 - \frac{15}{152}(16 - \sigma^2)\,k \quad \text{as} \quad x \to +\infty\,, \tag{25}$$

where $k = \sqrt{-2S}$. Expression (25) is valid only for $S < 0$ from Table 1. The other solutions, corresponding to $S > 0$ from Table 1, are periodic and, moreover, are singular at $\xi = \pm\pi/2$ and, consequently, do not tend to a limit as $x \to \pm\infty$.

The particular solutions obtained were used to test a numerical algorithm in mathematical modeling of physical processes described by Eq. (1) for arbitrary parameters α, β, γ.

Numerical Modeling of Nonlinear Waves Described by Eq. (1). For this, use is made of an implicit difference scheme of the following form with order of approximation $O(\tau) + O(h^2)$:

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{4h}\left[(u_{j+1}^n)^2 - (u_{j-1}^n)^2\right] +$$

$$+ \frac{\alpha}{2h^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + \frac{\beta}{4h^3}(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}) +$$

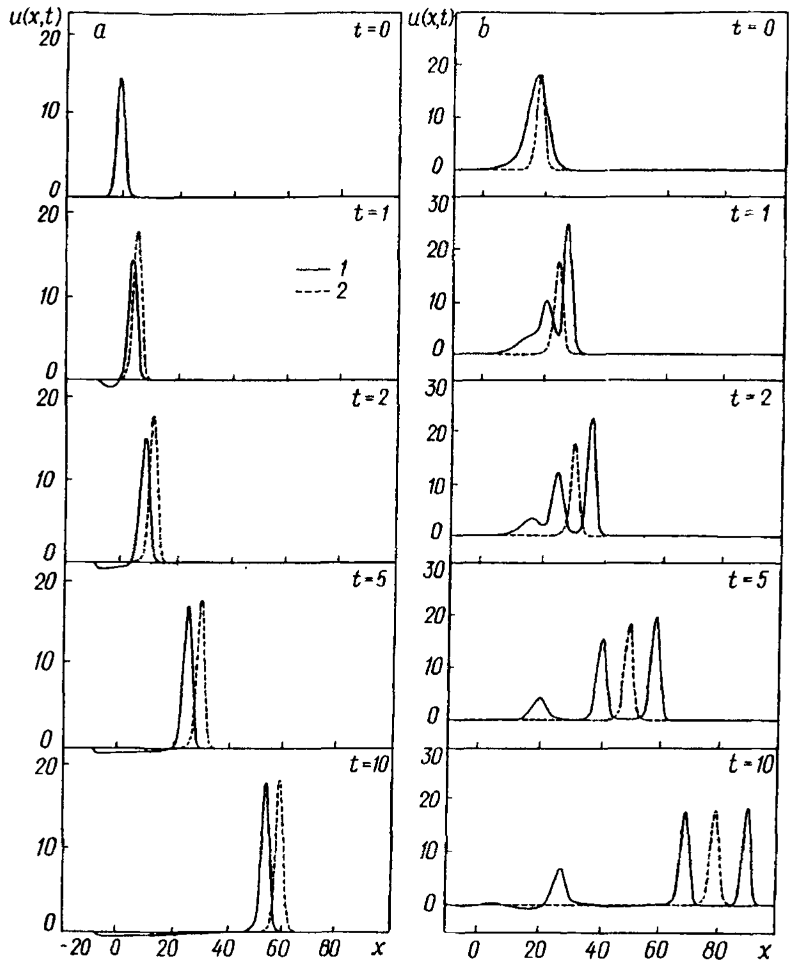

Fig. 2. Evolution of a wave specified at $t = 0$ by the expression $u(x) = A \cosh^{-2}(kx/2)\{1 - \tanh(kx/2)\}$: a) $A = 12$, $k = 1$; b) $A = 15$, $k = 0.4$; dashed line – solitary wave for $\sigma = 4$, $A = 15$, $k = 1$.

$$+ \frac{\gamma}{2h^4}\left(u_{j+2}^{n+1} - 4u_{j+1}^{n+1} + 6u_j^{n+1} - 4u_{j-1}^{n+1} + u_{j-2}^{n+1}\right) = 0. \tag{26}$$

The convergence of the numerical solution obtained by scheme (26) to an accurate solution of Eq. (1) depends on $\sigma = \beta/\sqrt{\alpha\gamma}$. For instance, for $\sigma = 0$ difference scheme (26) is applicable for $\tau \le h^2$, while for $\sigma = 4$ the constraint is more rigorous: $\tau \le h^3$.

Numerical modeling of the propagation of a wave specified at $t = 0$ by the expression

$$u(x) = 15\cosh^{-2}(x/2)\left\{1 - \tanh(x/2)\right\}, \tag{27}$$

showed that for $\sigma = 4$ it maintains its shape and propagates with the velocity $C = 6$. Its profile coincides with the analytical solution from Table 2 for $\sigma = 4$, $C = 6$ within the entire period of calculation. We investigated the interaction of solitary wave (27) with other disturbances. Figure 1 illustrates the interaction of this wave with a disturbance that has the form $u(x) = 15\cosh^{-2}\{8(x - 3)\}$ at $t = 0$. As is seen, the solitary wave overtakes the disturbance, interacts with it, and continues to propagate with the same velocity without changing its shape but with a change in the phase compared to propagation without collision. Thus, solitary wave (27) moves with a constant velocity without changing its shape and interacts elastically with other disturbances, and consequently it is a classical soliton.

Thus, Eq. (1) with nonzero α, β, γ has a soliton solution, but it is unique for a given set of parameters $\sigma = \beta/\sqrt{\alpha\gamma}$ since the other disturbance changes with time, its amplitude increases, and for sufficiently large times of

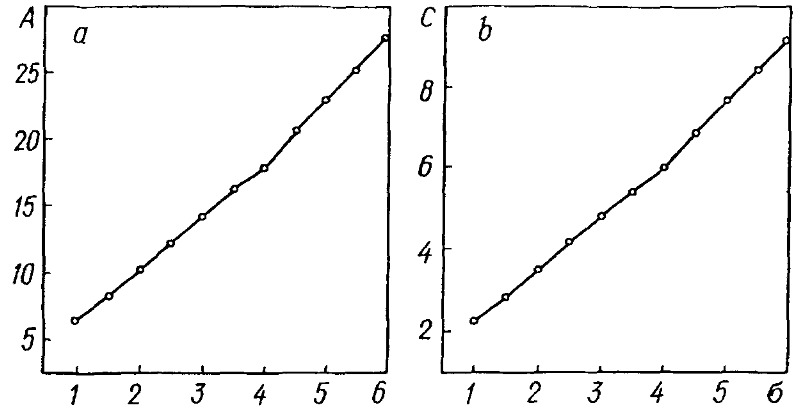

Fig. 3. Amplitude A and velocity C of the solitary wave as a function of σ: a) $A(\sigma)$; b) $C(\sigma)$.

calculation the disturbance acquires the form (27). However, the process of growth of the amplitude of the solitary wave does not depend on the interaction.

If, as the initial condition, we take a solitary wave with an amplitude smaller than (27), this wave will grow with time until its amplitude reaches the amplitude of solitary wave (27) (see Fig. 2a). On the other hand, if at the initial moment the amplitude of the wave is larger than that of the soliton solution corresponding to the given σ, then this wave will break into parts that over time, some earlier, some later, grow to wave (27).

A similar situation is observed if at the initial moment of time the width of the wave is larger than the width of the soliton corresponding to the given σ. Figure 2b illustrates the evolution, for $\sigma = 4$, of the solution

$$u(x) = 15 \cosh^{-2}(0.4x/2) \{1 - \tanh(0.4x/2)\},$$

This wave also breaks into parts, thus leading to formation of three waves, two of which have the form (27). The dashed curve represents the exact solution of Eq. (1) for $\sigma = 4$.

It should be noted that this picture is observed for various σ: all waves that exist in the system, transforming in some way, tend to acquire the shape of the soliton corresponding to the given particular σ.

We investigated the behavior of a wave as a function of σ. It is turned out that the amplitude of a soliton increases with σ, the velocity of propagation of the wave also increases. Figure 3 shows the amplitude and velocity of a soliton as a function of σ for zero boundary conditions.

To sum up, analytical solutions of Eq. (1) are obtained in form of solitary waves. With the aid of difference scheme (26) the interaction of a solitary wave with other solutions is investigated. The formation of solitary waves at different σ is considered. Dependences of the amplitude and velocity of a solitary wave on the parameter σ are determined.

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NOTATION

$u(x, t)$, function characterizing the deviation of the displacement, temperature, concentration, etc. from equilibrium; α, β, γ, constant coefficients of intensification, dispersion, and dissipation, respectively; $\sigma = \beta/\sqrt{\alpha\gamma}$, dimensionless combination of them; x, coordinate; t, time; u_j^{n+1}, value of u at $x_j = jh$, $t^{n+1} = (n+1)\tau$; τ, time step; h, coordinate step.

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